JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 39, 742–760 (1972)

Axisymmetric Inertial Oscillations of a Rotating Fluid

JOEL N. FRANKLIN

California Institute of Technology, Pasadena, California 91109

Submitted by Peter D. Lax

1. INTRODUCTION

In the paper [1], V. Barcilon uses an ingenious transformation of coordinates and separation of variables to solve the following problem: Find the numbers $\lambda > 0$ for which the hyperbolic equation

$$u_{\xi\xi} - \lambda^2 u_{\eta\eta} = 0 \tag{1}$$

has solutions $u \not\equiv 0$ in the unit circle, with $u = 0$ on the boundary. Barcilon also presents the history of this problem and its relation to fluid mechanics. The problem dates to Lord Kelvin (1880) and to Poincaré (1885). Other contributors referenced in [1] are Bryan (1889), Cartan (1922), Burgin and Duffin (1939), Høiland (1962), Greenspan (1964), Wood (1966) and Aldridge and Toomre (1969).

Here we shall solve the problem (1) for the general ellipse, for the triangle, and for the strip. We shall also find the eigenvalue $\lambda = 1$ for an annulus in which the ratio of the radii is $\geqslant \sqrt{2}$. The entire paper is based on the elementary method of characteristics.

2. METHOD OF CHARACTERISTICS FOR SIMPLY CONNECTED REGIONS

Let u be a solution of a hyperbolic equation

$$a\frac{\partial^2 u}{\partial \xi^2} + 2b\frac{\partial^2 u}{\partial \xi\, \partial \eta} + c\frac{\partial^2 u}{\partial \eta^2} = 0 \tag{1}$$

in a region D_0, where a, b, and c are constants, with $b^2 > ac$. A nonsingular linear transformation transforms the equation (1) into

$$\frac{\partial^2 u}{\partial x\, \partial y} = 0 \tag{2}$$

and maps D_0 into a region, D. In this section we shall suppose that D_0, and hence D, are simply connected.

Let P_0 be a fixed point in D. Then

$$u(P) = f(P) - g(P), \tag{3}$$

where

$$\begin{aligned} f(P) &= u(P_0) + \int_{P_0}^{P} u_x \, dx, \\ g(P) &= -\int_{P_0}^{P} u_y \, dy. \end{aligned} \tag{4}$$

The integrals defining $f(P)$ and $g(P)$ are independent of the path connecting P_0 to P because $u_{xy} = 0$. In fact, the integrals around closed circuits in D are

$$-\oint u_x \, dx = \iint u_{xy} \, dx \, dy = \oint u_y \, dy.$$

Note that the single-valuedness of f and of g depends on the simple connectivity of D.

In (3) we have

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0. \tag{5}$$

We shall call any identity $u = f - g$ in which (5) holds, a *standard form* for a function u solving $u_{xy} = 0$. *If $u = f_1 - g_1 = f_2 - g_2$ are two standard forms, then*

$$f_2(P) = f_1(P) + c, \qquad g_2(P) = g_1(P) + c, \tag{6}$$

where c is constant.

Proof.

$$\frac{\partial u}{\partial x} = \frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial g_1}{\partial y} = -\frac{\partial g_2}{\partial y}.$$

Hence,

$$(f_1 - f_2)_x = 0 = (g_1 - g_2)_y .$$

But (5) implies that

$$(f_1 - f_2)_y = 0 = (g_1 - g_2)_x .$$

The equations (5) do *not* imply that $f(P) =$ a function of x, and that $g(P) =$ a function of y. They only imply that f and g remain constant on, respectively, vertical or horizontal segments lying entirely in D. For example, in the simply connected domain given in Fig. 1.1 neither does f have to be a function of x, nor does g have to be a function of y. However, if every vertical

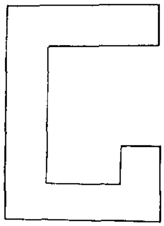

FIGURE 1.1

segment connecting points of D lies in D, then f is a function of x; if every horizontal segment connecting points of D lies in D, then g is a function of y.

Let $u_{xy} = 0$ in D, and let $u = 0$ on the boundary, ∂D. Then, if $u = f - g$ is a standard form in $\bar{D}$, the parts f and g are identical on the boundary. We define

$$h(P) \equiv f(P) = g(P) \qquad \text{for } P \in \partial D. \tag{7}$$

If $u \equiv 0$ *in* D, *then* $h(P) \equiv$ const *for* $P \in \partial D$. The proof is that the solution $u \equiv 0$ of $u_{xy} = 0$ has a standard form $u = f_0(P) - g_0(P)$, where $f_0 \equiv g_0 \equiv 0$. Hence, if $0 \equiv f - g$ is another standard form, we must have $f \equiv g \equiv c$, a constant, by (6).

The converse is false. For example, the function $u = \sin x$ solves $u_{xy} = 0$ in the strip $0 < x < \pi$, and u vanishes on the boundary. Here $f = \sin x$ and $g = 0$. The boundary-function $h(P)$ is identically zero although $u \not\equiv 0$ in the strip.

If the line $x = x_0$ and the line $y = y_0$ through each point (x_0, y_0) of D intersect the boundary ∂D in finite points, we call D *xy-bounded.* For example, the halfplane $3y - x > 0$ is xy-bounded, but the region $y > 0$ is not. Every bounded region is xy-bounded. The strip discussed in the last paragraph is not xy-bounded.

LEMMA. *Let the region D be simply connected and xy-bounded. Let* $u_{xy} = 0$ *in D and* $y = 0$ *in* ∂D. *Let* $u = f - g$ *be a standard form, and let* $h(P)$ *be the common value of f and g for* $P \in \partial D$. *Then* $u \equiv 0$ *in D if and only if* $h(P) \equiv$ const.

Proof. We have already proven that $u \equiv 0$ in D implies $h \equiv$ const. Conversely, suppose $h(P) \equiv c$. If $Q \in D$, then $u(Q) = f(Q) - g(Q)$. Let the vertical and the horizontal lines through Q intersect the boundary in points X and Y. In each case we take a nearest possible boundary point. Then the vertical and horizontal segments $\overline{QX}$ and $\overline{QY}$ lie in D. Hence,

$$f(Q) = f(X) = c, \qquad g(Q) = g(Y) = c.$$

Therefore, $u(Q) = c - c = 0$. ∎

If $u_{xy} = 0$ in D, $u = 0$ on ∂D, and $u \not\equiv 0$ in D, we call u an *eigenfunction*. We call $h(P)$ its *boundary-function* if h is defined by (7). The boundary function is unique apart from an additive constant, since the parts, f and g, of a standard form are unique apart from an additive constant, as in (6). We will now state the conditions which insure that a function $b(P)$ prescribed on ∂D is a boundary function $h(P)$ for some eigenfunction u.

THEOREM. *Let the region D be simply connected and xy-bounded. Let the differentiable function $b(P)$ be defined on ∂D. Then $b(P)$ is a boundary-function $h(P)$ for an eigenfunction u if and only if $b(P)$ is not constant, and $b(P_1) = b(P_2)$ for every vertical or horizontal segment $\overline{P_1P_2}$ lying in the closure of D and connecting two boundary points.*

Proof. If $b(P)$ satisfies the hypotheses, we can construct an eigenfunction u as follows: If $Q \in D$, since D is xy-bounded, there is a boundary-point X such that $\overline{QX}$ is a vertical segment in $\bar{D}$. Define $f(Q) \equiv b(X)$. This definition is unambiguous, for if X' is another such boundary-point, then $\overline{XX'}$ is a vertical segment lying in $\bar{D}$ and connecting two boundary points; therefore, $b(X) = b(X')$. Similarly, there is a boundary-point Y such that $\overline{QY}$ is a horizontal segment in $\bar{D}$, and the definition $g(Q) \equiv b(Y)$ is unambiguous. Moreover, $f_y = g_x = 0$.

Hence, $u \equiv f - g$ is a solution of $u_{xy} = 0$ in D. For $Q \in \partial D$, we may take $X = Y = Q$; hence $u = 0$ on ∂D. The common value of $f(P)$ and $g(P)$ on the boundary is $b(P)$, and $b(P)$ is assumed to be nonconstant. The lemma now implies that u is an eigenfunction, and that $h(P) = b(P)$ is its boundary function.

Conversely, if u is an eigenfunction with a boundary-function $h(P)$, the lemma implies that $h \not\equiv$ const. Moreover, if $\overline{P_1P_2}$ is a vertical or horizontal segment lying in $\bar{D}$ and connecting two boundary points, then $h(P_1) = h(P_2)$ because f is constant on the segment if it is vertical, whereas g is constant on the segment if it is horizontal. Therefore, $h(P)$ satisfies the hypotheses required of $b(P)$. ▮

Smoothness of the boundary-function

We will henceforth assume that $h(P)$ is piecewise smooth on ∂D, i.e., continuous andpiecewise continuously differentiable. This is a mild assumption, since

$$h(P) = \text{const} + \int_{P_0}^{P} u_x \, dx = \text{const} - \int_{P_0}^{P} u_y \, dy. \tag{8}$$

Thus, continuity of u_x and u_y in $\bar{D}$ would imply the continuous differentiability of h.

3. The Triangle

Let D_0 be a nondegenerate triangle in the ξ, η-plane. Let u satisfy any hyperbolic equation (2.1) in D_0, with $u = 0$ on ∂D_0. Then we will prove that $u \equiv 0$. In particular, if $u_{\xi\xi} - \lambda^2 u_{\eta\eta} = 0$ in D_0, with $u = 0$ on ∂D_0, and if $\lambda > 0$, then $u \equiv 0$.

In the characteristic variables, the partial differential equation becomes $u_{xy} = 0$, and D_0 is mapped into a nondegenerate triangle, D, in the x, y-plane. Let V_1, V_2, V_3 be the vertices of D. Let (a_i, b_i) be the x, y-coordinates of V_i. We may assume

$$a_1 < a_3 \qquad \text{and} \qquad a_1 \leqslant a_2 \leqslant a_3 . \tag{1}$$

We may also assume $b_1 \leqslant b_3$; for if $b_1 > b_3$, the mapping $x' = x, y' = -y$ maps u into a solution of $u_{x'y'} = 0$, while the vertices V_i' satisfy the requirements (1) and $b_1' \leqslant b_3'$.

If u were an eigenfunction, let $h(P)$ be its boundary function. We will prove that either $h(P) \equiv h(V_3)$, a constant, or $h(P) \equiv h(V_1)$, a constant, which contradicts the requirement that $h(P)$ be nonconstant. If $P = (x, y)$, let P_1 be the point on $\overline{V_1V_3}$ with coordinates (x_1, y_1), where $x_1 = x$. ($P = P_1$ if $P \in \overline{V_1V_3}$).

If the edge $\overline{V_1V_3}$ is horizontal, we conclude at once

$$h(P) = h(P_1) = h(V_3).$$

Therefore, we may now assume $b_1 < b_3$.

We need to distinguish two cases. First suppose that V_2 lies *below* the edge $\overline{V_1V_3}$, as in Fig. 3.1.

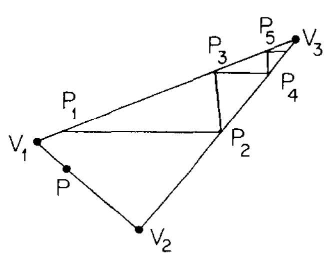

Figure 3.1

If $x_1 < a_3$, we construct an infinite sequence of boundary points $P_2, P_3, \ldots$ as follows. If n is even, P_n is the boundary point reached by moving to the right from P_{n-1}. If n is odd, P_n is the point on $\overline{V_1V_3}$ reached by moving up from P_{n-1}. Let $P_n = (x_n, y_n)$. Then

$$x = x_1 < x_2 = x_3 < x_4 = x_5 < \cdots .$$

The monotone nondecreasing sequence x_n has a limit, $l \leqslant a_3$. To prove that $l = a_3$, let $\varphi(x)$ be the continuous function which maps x_1 into x_3, or x_3 into x_5, etc. Then $l = \varphi(l)$. But $t < \varphi(t)$ if $a_1 \leqslant t < a_3$. Therefore, $l = a_3$.

Since every segment $\overline{P_n P_{n+1}}$ is vertical or horizontal, the boundary-function $h(P)$ satisfies

$$h(P) = h(P_1) = h(P_2) = \cdots .$$

Since $x_n \to a_3$ as $n \to \infty$, $P_n \to V_3$. By the continuity of the boundary function,

$$h(P) = h(P_n) = \lim_{n\to\infty} h(P_n) = h(V_3) = \text{const.}$$

Hence, the lemma in Section 2 implies that no eigenfunction exists.

In the other case, V_2 lies *above* $\overline{V_1 V_3}$, as in Fig. 3.2. Let $P = (x, y) \in \partial D$.

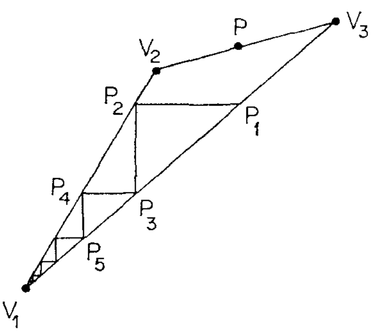

FIGURE 3.2

Let $P_1 = (x_1, y_1) \in \overline{V_1 V_3}$, with $x_1 = x$. If $x_1 > a_1$, form P_2 by moving left, then P_3 by moving down, then P_4 by moving left, etc. Then

$$x = x_1 > x_2 = x_3 > x_4 = x_5 > \cdots .$$

Exactly as before, we conclude that

$$x_n \to a_1, \qquad P_n \to V_1, \qquad h(P) = h(P_n) \to h(V_1) = \text{const.}$$

Again the lemma implies that no eigenfunction exists.

4. The Strip

Let D_0 be the strip

$$c_1 < p\xi + q\eta < c_2, \tag{1}$$

where $p,\ q \neq 0, 0$. We seek the numbers $\lambda > 0$ which permit solutions $u \not\equiv 0$ to the boundary-value problem

$$u_{\xi\xi} - \lambda^2 u_{\eta\eta} = 0 \quad \text{in} \quad D_0, \qquad u = 0 \quad \text{in} \quad \partial D_0. \tag{2}$$

We will show that *every* $\lambda > 0$ permits solutions $u \not\equiv 0$. This problem provides a simple illustration of the theorem.

The characteristic variables

$$x = \lambda\xi - \eta, \qquad y = \lambda\xi + \eta \tag{3}$$

transform (1) and (2) into

$$D : 2\lambda c_1 < (p - \lambda q)\, x + (p + \lambda q)\, y < 2\lambda c_2\,, \tag{4}$$

$$u_{xy} = 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{in} \quad \partial D. \tag{5}$$

If $p \pm \lambda q = 0$, then D is not xy-bounded; and the theorem is inapplicable. But these cases are trivial. If $p + \lambda q = 0$, the general solution is $u = f(x)$ provided only that $f = 0$ on the boundary lines

$$x = 2\lambda c_i/(p - \lambda q) \qquad (i = 1, 2).$$

If, instead, $p - \lambda q = 0$, we have $u = -g(y)$ if $g = 0$ on the lines

$$y = 2\lambda c_i/(p + \lambda q) \qquad (i = 1, 2).$$

If the boundary lines of the strip do not lie in a characteristic direction, i.e., if $p \pm \lambda q \neq 0$, then D is xy-bounded, and the theorem is applicable. If $P = (x, y) \in \partial D$, the identity $h(P) \equiv f(x)$ for a boundary function automatically implies that $h(P_1) = h(P_2)$ if $\overline{P_1P_2}$ is vertical in the x, y-plane. It remains only to require that $h(P_1) = h(P_2)$ if $\overline{P_1P_2}$ is horizontal. Suppose that $P_1 = (x, y)$ lies on the boundary-line

$$2\lambda c_1 = (p - \lambda q)\, x + (p + \lambda q)\, y.$$

If $P_2 = (x', y)$ has the same ordinate, but lies on the other boundary line, then

$$2\lambda c_2 = (p - \lambda q)\, x' + (p + \lambda q)\, y.$$

Hence,

$$x' = x + 2\lambda(c_2 - c_1)/(p - \lambda q).$$

The requirement $h(P_1) = h(P_2)$ means that $f(x) = f(x')$.

Hence, the only requirements on $h(P) = f(x)$ are that $f(x)$ be nonconstant and that $f(x)$ have the period $2\lambda(c_2 - c_1)/(p - \lambda q)$. To find the corresponding eigenfunction, u, let $Q = (x, y)$ lie inside the strip D. Then

$$u(Q) = h(X) - h(Y), \tag{6}$$

where X is either of the two boundary points with abscissa x, and Y is either of the two boundary points with the ordinate y. Therefore,

$$h(X) = f(x), \qquad h(Y) = f(x_1),$$

where the boundary-point $Y = (x_1, y)$ satisfies

$$2\lambda c_1 = (p - \lambda q)\, x_1 + (p + \lambda q)\, y.$$

Now (6) yields

$$u = f(x) - f([2\lambda c_1 - (p + \lambda q)\, y]/[p - \lambda q]). \tag{7}$$

The periodicity of $f(x)$ guarantees that $u = 0$ on both boundary lines. In the original variables, ξ and η, (7) becomes

$$u = f(\lambda\xi - \eta) - f([2\lambda c_1 - (p + \lambda q)\,(\lambda\xi + \eta)]/[p - \lambda q]).$$

5. The Ellipse

Let D_0 be the ellipse

$$f_{11}\xi^2 + 2f_{12}\xi\eta + f_{22}\eta^2 < 1, \tag{1}$$

where $F = (f_{ij})$ is positive definite. Let constants a, b, c be given, with $b^2 > ac$. We will determine whether there exists a solution $u \not\equiv 0$ to the boundary-value problem

$$au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} = 0 \quad \text{in} \quad D_0\,, \qquad u = 0 \quad \text{in} \quad \partial D_0\,. \tag{2}$$

In particular, we will find the eigenvalues $\lambda > 0$ for which

$$u_{\xi\xi} - \lambda^2 u_{\eta\eta} = 0 \quad \text{in} \quad D_0\,, \qquad u = 0 \quad \text{in} \quad \partial D_0 \tag{3}$$

admits solutions $u \not\equiv 0$. We shall show that the eigenvalues comprise a denumerable set $\{\lambda_{pq}\}$ which is dense in the set of positive numbers. Let p and q be relatively prime, positive integers, with $p < q$. Let $\Delta = f_{11} f_{22} - f_{12}^2$. Then

$$\lambda_{pq} = f_{22}^{-1} \left\{ \Delta^{1/2} \cot \frac{p\pi}{q} + \left(\Delta \cot^2 \frac{p\pi}{q} + f_{11} f_{22} \right)^{1/2} \right\}. \tag{4}$$

For *each* $\lambda = \lambda_{pq}$, the solutions $u \not\equiv 0$ of (3) are

$$u = \sum_{k=1}^{\infty} c_k [T_{kq}((\lambda\xi - \eta)\, J(\lambda)) - (-)^{kp}\, T_{kq}((\lambda\xi + \eta)\, J(-\lambda))], \tag{5}$$

where the c_k are arbitrary coefficients satisfying

$$0 < \sum_{1}^{\infty} | c_k | < \infty, \tag{6}$$

where $T_n(z)$ is the Chebyshev polynomial

$$T_n(\cos\theta) \equiv \cos n\theta,$$

and where

$$J(\lambda) \equiv \Delta^{1/2}(f_{11} + 2\lambda f_{12} + \lambda^2 f_{22})^{-1/2}. \tag{7}$$

The solution is a direct application of the method of characteristics. A nonsingular linear transformation

$$x = c_{11}\xi + c_{12}\eta, \qquad y = c_{21}\xi + c_{22}\eta \tag{8}$$

takes (1) and (2) into an ellipse

$$D : e_{11}x^2 + 2e_{12}xy + e_{22}y^2 < 1 \tag{9}$$

and into the boundary-value problem

$$u_{xy} = 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{in} \quad \partial D. \tag{10}$$

The boundary, ∂D, of the ellipse (9) may be written in the parametric form

$$x = \rho\cos\theta, \qquad y = \sigma\cos(\theta - \alpha) \qquad \text{for } 0 \leqslant \theta < 2\pi, \tag{11}$$

where the constants ρ, σ, α satisfy

$$\rho > 0, \qquad \sigma > 0, \qquad 0 < \alpha < \pi. \tag{12}$$

If $\delta = e_{11}e_{22} - e_{12}^2$, then the constants ρ, σ, α are

$$\rho = \left(\frac{e_{22}}{\delta}\right)^{1/2}, \qquad \sigma = \left(\frac{e_{11}}{\delta}\right)^{1/2}, \qquad \alpha = \operatorname{arc\,cot}(-\delta^{-1/2}e_{12}). \tag{13}$$

If u is an eigenfunction, it has a nonconstant boundary function $h = h(\theta)$ where h has the period 2π. If $P = (x, y)$ lies on the boundary (11), we must have

$$h(\theta) = h(\theta_X) = h(\theta_Y), \tag{14}$$

where X and Y are the *other* boundary points with, respectively, the same abscissa and the same ordinate as (x, y). Therefore, by (11),

$$\theta_X \equiv -\theta (\text{mod } 2\pi), \qquad \theta_Y \equiv 2\alpha - \theta (\text{mod } 2\pi). \tag{15}$$

We assume that $h(\theta)$ has an absolutely convergent Fourier series

$$h(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{16}$$

We may suppose $a_0 = 0$ because, if a constant is subtracted from a boundary function, a new boundary function results which defines the same eigenfunction. Moreover, all $b_n = 0$, since

$$h(\theta) = h(\theta_X) = h(-\theta).$$

The only remaining requirement is $h(\theta) = h(\theta_Y)$

$$\sum_{n=1}^{\infty} a_n \cos n\theta = \sum_{n=1}^{\infty} a_n \cos n(2\alpha - \theta) \tag{17}$$

with some $a_n \neq 0$. This must be an *identity* in θ.

If a_k is a particular, nonzero coefficient, the identity (17) implies that $\cos 2k\alpha = 1$, and hence $2k\alpha = 2j\pi$, where j is an integer. Since $0 < \alpha < \pi$, we require $1 \leqslant j < k$. If $j/k = p/q$, where *p and q are relatively prime, with* $1 \leqslant p < q$, then

$$\alpha = \frac{p\pi}{q}. \tag{18}$$

This solves the problem. *There exists a solution $u \not\equiv 0$ to the boundary-value problem if and only if the ellipse satisfies* (18). By the identity (13) for α, the criterion (18) becomes

$$\frac{-e_{12}}{\sqrt{(e_{11}e_{22} - e_{12}^2)}} = \cot \frac{p\pi}{q} \qquad (1 \leqslant p < q; (p, q) = 1). \tag{19}$$

If α has the required form (18), we can find the associated eigenfunctions $u \not\equiv 0$. By (17), $a_n = 0$ unless $2n\alpha$ is a multiple of 2π. Since $(p, q) = 1$, $a_n = 0$ unless n is a multiple of q. If $c_k \equiv a_{kq}$, (16) yields

$$h(\theta) = \sum_{k=1}^{\infty} c_k \cos kq\theta. \tag{20}$$

If $h(\theta)$ is required to have an absolutely convergent Fourier series, the coefficients c_k are arbitrary members satisfying (6). As in Section 2, if $Q = (x, y) \in D$, let X and Y be boundary points with, respectively, the abscissa x and the ordinate y. Then

$$u = h(\theta_X) - h(\theta_Y). \tag{21}$$

By (11),

$$x = \rho \cos \theta_X , \qquad y = \sigma \cos(\theta_Y - \alpha). \tag{22}$$

Hence,

$$\cos kq\theta_X = T_{kq}(x/\rho), \tag{23}$$

$$\cos kq(\theta_Y - \alpha) = T_{kq}(y/\sigma). \tag{24}$$

Since $kq\alpha = kq(p\pi/q) = kp\pi$, (24) yields

$$(-)^{kp} \cos kq\theta_Y = T_{kq}(y/\sigma). \tag{25}$$

Now (20), (21), (23), and (25) imply

$$u = \sum_{k=1}^{\infty} c_k[T_{kq}(x/\rho) - (-)^{kp}\, T_{kq}(y/\sigma)]. \tag{26}$$

For fixed $\alpha = p\pi/q$, formula (26) gives *all* the eigenfunctions whose boundary functions $h(\theta)$ have absolutely convergent Fourier series. Moreover, if $0 < \sum | c_k' | < \infty$, the eigenfunction u' obtained by replacing the coefficients c_k by c_k', is different from u unless *all* $c_k' = c_k$ $(k = 1, 2,...)$, since the associated boundary functions differ by a constant only if their Fourier coefficients multiplying $\cos kq\theta$ are identical. Thus, *if the criterion* (18) *is satisfied, there exists a denumerable set of linearly independent eigenfunctions.*

Now we will obtain the asserted formulas (4), (5), (7) solving the particular boundary-value problem (3). Let (8) take the form

$$x = \frac{1}{2}\left(\xi - \frac{1}{\lambda}\eta\right), \qquad y = \frac{1}{2}\left(\xi + \frac{1}{\lambda}\eta\right). \tag{27}$$

The original ellipse D_0 in (1) becomes the ellipse D, as in (9), where

$$\begin{aligned} e_{11} &= f_{11} - 2\lambda f_{12} + \lambda^2 f_{22} , \\ e_{12} &= f_{11} - \lambda^2 f_{22} , \\ e_{22} &= f_{11} + 2\lambda f_{12} + \lambda^2 f_{22} . \end{aligned} \tag{28}$$

Since $\delta = \det E$ and $\Delta = \det F$, we have

$$\delta = 4\lambda^2\Delta. \tag{29}$$

The criterion (19) becomes

$$\frac{\lambda^2 f_{22} - f_{11}}{2\lambda\Delta^{1/2}} = \cot \frac{p\pi}{q} \qquad (1 \leqslant p < q; (p, q) = 1). \tag{30}$$

This equation uniquely determines the positive eigenvalue (4). All the eigenfunctions, u, belonging to the eigenvalue $\lambda = \lambda_{pq}$ are given by (26). By (13), (27), (28) and (29) the arguments x/ρ and y/σ become

$$\begin{aligned} \frac{x}{\rho} &= \frac{1}{2}\left(\xi - \frac{1}{\lambda}\eta\right)\delta^{1/2}e_{22}^{-1/2} = (\lambda\xi - \eta)\,J(\lambda), \\ \frac{y}{\sigma} &= \frac{1}{2}\left(\xi + \frac{1}{\lambda}\eta\right)\delta^{1/2}e_{11}^{-1/2} = (\lambda\xi + \eta)\,J(-\lambda), \end{aligned} \tag{31}$$

where $J(\lambda)$ is defined in (7). Now (26) yields the asserted form (5).

6. Multiply Connected Regions

The generalities in Section 2 must be modified if D is multiply connected. For instance, let D be doubly connected, as in Fig. 6.1. Let $u_{xy} = 0$ in D. Let P_0 be a fixed point in D. Let $\partial D = C_1 \cup C_2$. If P is any point in D, then

$$u = u(P_0) + \int_{P_0}^{P} (u_x\, dx + u_y\, dy), \tag{1}$$

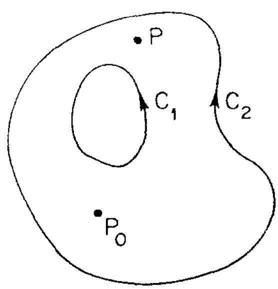

Figure 6.1

where the integral is path independent. Since $u_{xy} = 0$, the integral

$$\int_{P_0}^{P} u_x\, dx \tag{2}$$

has the same value for all homotopic arcs joining P_0 to P. But different values result from nonhomotopic arcs if the constant

$$\gamma_1 \equiv \oint_{C_1} u_x\, dx \tag{3}$$

is nonzero. The same remarks apply to the integral

$$\int_{P_0}^{P} u_y \, dy. \tag{4}$$

Since u is single valued,

$$-\oint_{C_1} u_y \, dy = \oint_{C_1} u_x \, dx = \gamma_1 . \tag{5}$$

An example will show that the integrals (2) and (4) really may be multivalued. Let D be the region between the square with vertices $(\pm 1, \pm 1)$ and the square with vertices $(\pm 2, \pm 2)$, as in Fig. 6.2. For $-2 \leqslant z \leqslant 2$ define

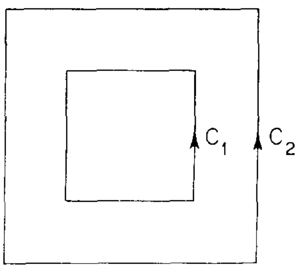

FIGURE 6.2

a function $\varphi(z)$ as follows: Let $\varphi \equiv 0$ for $-2 \leqslant z \leqslant -1$. Let φ increase from 0 to 1 as z increases from -1 to $+1$. Let $\varphi \equiv 1$ for $1 \leqslant z \leqslant 2$. At the points $z = \pm 1$, require that φ have all derivatives $=0$. Now define $u(x, y)$ in D as follows:

$$\begin{aligned}
&u = \varphi(x) && \text{if} && -2 \leqslant y \leqslant -1 && \text{(the bottom)},\\
&u = \varphi(-y) && && 1 \leqslant x \leqslant 2 && \text{(the right)},\\
&u = \varphi(-x) && \text{if} && 1 \leqslant y \leqslant 2 && \text{(the top)},\\
&u = \varphi(y) && \text{if} && -2 \leqslant x \leqslant -1 && \text{(the left)}.
\end{aligned} \tag{6}$$

Note that u is defined consistently in the four overlapped squares. The function u is a single-valued solution of $u_{xy} = 0$ because, in the neighborhood of any point, it is either a function of x alone or of y alone. Moreover,

$$\gamma_1 = \oint_{C_1} u_x \, dx = -\oint_{C_1} u_y \, dy = 2. \tag{7}$$

Now consider the general case of an n-tuply connected region, D, where $n \geqslant 2$. For simplicity, suppose D is bounded. Let D have the inner boundary curves $C_1, \dots, C_{n-1}$ and the outer boundary curve C_n. Let each C_i be a simple, closed curve. Let the curves be oriented so that, if C_i is traversed in its positive orientation, points of D appear on the left if $i = n$, but on the right if $i < n$. For $i = 1, \dots, n-1$, let C_i be connected to C_n by a branch cut, Γ_i, in D. Let no two branch cuts intersect each other. On each cut, Γ_i, define a $+$ side and a $-$ side, so that, if C_i is traversed positively, C_i begins on the $-$ side of Γ_i and ends on the $+$ side. For $n = 3$, Fig. 6.3 illustrates these conditions.

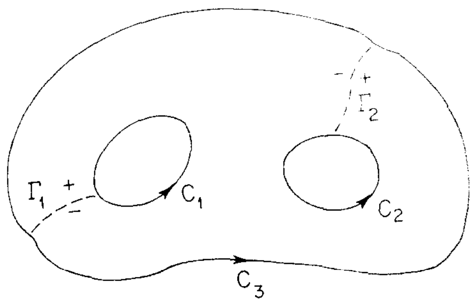

FIGURE 6.3

Let D_1 be the *simply* connected region

$$D_1 = D - \Gamma_1 - \Gamma_2 - \cdots - \Gamma_{n-1}. \tag{8}$$

Let P_0 be a fixed point in D_1. Let $u_{xy} = 0$ in D. Since D_1 is simply connected, u has a standard form

$$u(P) = f(P) - g(P), \tag{9}$$

where $f(P)$ and $g(P)$ are *single-valued* functions in D_1 for which

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0. \tag{10}$$

For example, if the path of integration does not cross a cut, we may let

$$\begin{aligned} f(P) &= u(P_0) + \int_{P_0}^{P} u_x \, dx, \\ g(P) &= -\int_{P_0}^{P} u_y \, dy. \end{aligned} \tag{11}$$

Every other standard form for u is obtained by adding a common constant to the two functions defined in (11).

We can now describe the behavior of the standard form as P jumps over one of the cuts, Γ_i. Let

$$\gamma_i = \int_{C_i} u_x\,dx = -\int_{C_i} u_y\,dy \qquad (i = 1,\ldots, n). \tag{12}$$

By the choice of the orientations of $C_1,\ldots, C_{n-1}, C_n$,

$$\gamma_n = \gamma_1 + \cdots + \gamma_{n-1}. \tag{13}$$

For $i = 1,\ldots, n-1$, the formulas (11) show that γ_i is the jump made by $f(P)$ or $g(P)$ from the negative side of Γ_i to its positive side: If $P \in \Gamma_i$, then

$$\gamma_i = f(P_+) - f(P_-) = g(P_+) - g(P_-). \tag{14}$$

The jump γ_i is the same at *all* points $P \in \Gamma_i$.

Let $u_{xy} = 0$ in D, and now suppose $u = 0$ in ∂D. We wish to define a boundary function, $h(P)$, for u. If $i < n$, let C_i' be the cut curve formed by removing from C_i the single point at which C_i joins Γ_i; if $i = n$, let C_n' be set formed by removing from C_n the $n-1$ points joining C_n to $\Gamma_1,\ldots, \Gamma_{n-1}$. In D_1 let u have the standard form (9), (11). Assuming that the integrals (11) converge as $P \to C_i'$, we may define $f(P)$ and $g(P)$ even for $P \in C_i'$. *Since* $u = 0$ in ∂D, *the parts* $f(P)$ *and* $g(P)$ *have a common value,* $h(P)$, *for* $P \in C_i'$. Thus we may define the boundary function

$$h(P) = f(P) = g(P) \qquad \text{for } P \in C_1' \cup \cdots \cup C_n'. \tag{15}$$

We will now describe the requirements on a boundary function. Let P and Q be points on $\cup\, C_i'$, which is simply ∂D minus the $2(n-1)$ points terminating the cuts. Let the segment $\overline{PQ}$ be vertical or horizontal, and suppose that $\overline{PQ}$ lies entirely in the original, uncut region, D. If this segment does not cross a cut, either f or g is constant on the segment, by (10), so $h(P) = h(Q)$. If $\overline{PQ}$ crosses one or more cuts, we must use the jump relations (14). Let ν_i be the integer which equals the number of times which the directed segment $\overline{PQ}$ crosses Γ_i from the $+$ to the $-$ side, minus the number of crossings from the $-$ to the $+$ side. Thus, ν_i equals 1, 0, or -1. Now (14) implies

$$h(P) - h(Q) = \nu_1\gamma_1 + \cdots + \nu_{n-1}\gamma_{n-1}. \tag{16}$$

If $u \equiv 0$ in D, the boundary function $h(P)$ is constant in $\cup\, C_i'$, since $f(P)$ and $g(P)$ are equal to a constant in the simply connected region D_1. Then, of course, all $\gamma_i = 0$.

If $u \not\equiv 0$ in D, but $u_{xy} = 0$ in D and $u = 0$ in ∂D, we call u an eigenfunction. *An eigenfunction is determined by its boundary function,* $h(P)$, as follows:

If $Q \in D_1$, let X and Y be nearest boundary points reached, respectively, by moving vertically and horizontally in D. The segments $\overline{QX}$ and $\overline{QY}$ may cross cuts, Γ_i, but not boundary curves, C_i. Since $u(Q) = f(Q) - g(Q)$, we only need to relate $f(Q)$ to $h(X)$ and to relate $g(Q)$ to $h(Y)$.

First suppose that neither X nor Y is one of the $2(n - 1)$ points at which the boundary curves C_i join the cuts Γ_j. Let ξ_i and η_i be, respectively, the integers which equal the number of times which the directed segments $\overline{QX}$ and $\overline{QY}$ cross Γ_i from the $+$ side to the $-$ side, minus the number of crossings from the $-$ side to the $+$ side. The jump conditions (14) now imply

$$\begin{aligned} f(Q) - f(X) &= \xi_1\gamma_1 + \cdots + \xi_{n-1}\gamma_{n-1}, \\ g(Q) - g(Y) &= \eta_1\gamma_1 + \cdots + \eta_{n-1}\gamma_{n-1}. \end{aligned} \tag{17}$$

Since $f(X) = h(X)$ and $g(Y) = h(Y)$, we deduce

$$u(Q) = h(X) - h(Y) + (\xi_1 - \eta_1)\,\gamma_1 + \cdots + (\xi_{n-1} - \eta_{n-1})\,\gamma_{n-1}. \tag{18}$$

This identity has been derived only for Q in D_1 and for X and Y in $\cup\, C_i'$. We will now show that, if limits are used, *the identity* (18) *determines* $u(Q)$ *for all* Q *in* D. We only need to show that the right-hand side of (18) is continuous as Q crosses a cut, Γ_i, or as X or Y crosses one of the $2(n - 2)$ points joining the cuts to the boundary curves. But this follows directly from (14).

In summary, a function $h(P)$ is the boundary function of an eigenfunction $u(Q)$ under the following conditions: $h(P)$ is defined as a smooth, nonconstant function for $P \in \cup\, C_i'$; for $i < n$, $h(P)$ jumps by γ_i as P jumps from Γ_i^- to Γ_i^+ on C_i' or on C_n; and h satisfies the identity (16) if P and Q are in $\cup\, C_i'$ and if the segment $\overline{PQ}$ is vertical or horizontal and lies in $\bar{D}$.

In Section 7, we will show that, if u is an eigenfunction for a circular annulus, then $\gamma_1 = 0$, and the boundary-function $h(P)$ is continuous on all of ∂D, even at the two points terminating the cut. Then $u(Q)$ has a standard form, $f(Q) - g(Q)$, in which $f(Q)$ and $g(Q)$ are single valued in the uncut, doubly connected region, D. One may now ask whether, in fact, these conditions exist for every doubly connected domain, D, admitting an eigenfunction. But Fig. 6.4 shows that there really are eigenfunctions for which $\gamma_1 \neq 0$. In this figure, D is doubly connected. The numbers indicate certain values taken by $h(P)$. Between the indicated values, $h(P)$ is required to vary smoothly and monotonely, except at the cut. The dots represent the cut, across which $h(P)$ jumps by $\gamma_1 = 4$. The continuous, single-valued eigenfunction is given by

$$u(Q) = h(X) - h(Y) - 4\xi_1, \tag{19}$$

where $\xi_1 = 1, 0,$ or -1, depending upon how or whether the vertical segment $\overline{QX}$ crosses the cut. This is an example of the identity (18) in which $\eta_1 = 0$

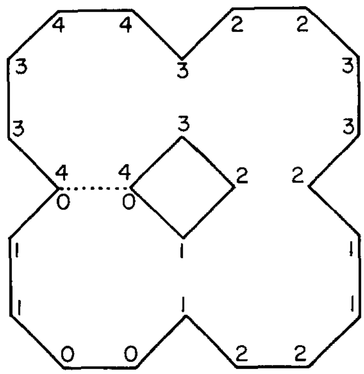

FIGURE 6.4

because the cut is horizontal. The eigenfunction $u(Q)$ in (19) takes values between -1 and $+1$.

7. The Annulus

In this section we will prove only that the boundary value problem,

$$u_{\xi\xi} - \lambda^2 u_{\eta\eta} = 0 \quad \text{in} \quad D_0, \qquad u = 0 \quad \text{in} \quad \partial D_0, \tag{1}$$

has a solution $u \not\equiv 0$ if $\lambda = 1$ and if D_0 is the annulus

$$D_0 : a^2 < \xi_2 + \eta^2 < 1, \tag{2}$$

where $0 < a \leqslant 1/\sqrt{2}$. We conjecture that there are other eigenvalues, λ.

If $\lambda = 1$, the 45° rotation

$$x = (\xi - \eta)/\sqrt{2}, \qquad y = (\xi + \eta)/\sqrt{2}, \tag{3}$$

transforms (1) and (2) into

$$u_{xy} = 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{in} \quad \partial D, \tag{4}$$

where $D = D_0$.

Define $b = (1 - a^2)^{1/2}$. In the interval $0 \leqslant x \leqslant b$ define $h(x)$ to be *any* nonconstant, twice-continuously differentiable function having the following properties:

(i) $h(x)$ has four continuous derivatives in a neighborhood of $x = 0$.

(ii) $h'(0) = h'''(0) = 0$.

(iii) $h(b) = h(0)$, $h'(b) = bh''(0)$, $h''(b) = h''(0) + \frac{1}{3} b^2 h^{(4)}(0)$.

Now define $h(x)$ in the rest of the interval $-1 \leqslant x \leqslant 1$ by the identities

$$\begin{aligned} h(x) &= h([x^2 - b^2]^{1/2}) \qquad \text{if} \quad b < x \leqslant 1, \\ h(x) &= h(-x) \qquad\qquad\quad \text{if} \quad -1 \leqslant x < 0. \end{aligned} \tag{5}$$

The properties (i), (ii), (iii) insure that the extension (5) of $h(x)$ produces a function which is twice-continuously differentiable on the interval $-1 \leqslant x \leqslant 1$, even at the points $x = 0, b, -b$. The assumption $a \leqslant 1/\sqrt{2}$ is used in (5) to guarantee that

$$0 < [x^2 - b^2]^{1/2} \leqslant b \qquad \text{if} \qquad b < x \leqslant 1.$$

An eigenfunction is now given by

$$\begin{aligned} u &= h(x) - h([1 - y^2]^{1/2}) \\ &= h((\xi - \eta)/\sqrt{2}) - h([1 - \tfrac{1}{2}(\xi - \eta)^2]^{1/2}). \end{aligned} \tag{6}$$

To show that (6) gives an eigenfunction, we need to show

$$u \not\equiv 0 \qquad \text{for} \qquad a^2 < x^2 + y^2 < 1, \tag{7}$$

$$u_{xy} = 0 \qquad \text{for} \qquad a^2 < x^2 + y^2 < 1, \tag{8}$$

$$u = 0 \qquad \text{if} \qquad x^2 + y^2 = 1, \tag{9}$$

$$u = 0 \qquad \text{if} \qquad x^2 + y^2 = a^2. \tag{10}$$

The relations (7)–(9) follow at once from the form of (6) and from the assumption that $h(x)$ is nonconstant. To prove (10), we deduce from (5)

$$h([1 - y^2]^{1/2}) = h([a^2 - y^2]^{1/2}) \qquad \text{if} \qquad 0 \leqslant y^2 \leqslant a^2.$$

Then, if $x^2 + y^2 = a^2$, (6) becomes

$$u = h([a^2 - y^2]^{1/2}) - h([1 - y^2]^{1/2}) = 0. \tag{11}$$

In this example, the boundary function h is smoothly defined, with no jump, on all of ∂D; and the standard form (6) consists of functions $f(x)$ and $g(y)$ which are single valued in the uncut annulus, D. Thus, if $a \leqslant r \leqslant 1$,

$$\gamma_1 = \gamma_2 = \int_{x^2+y^2=r^2} u_x \, dx = -\int u_y \, dy = 0. \tag{12}$$

This result for the annulus has a simple generalization:

LEMMA. *Let $u_{xy} = 0$ in a bounded, n-tuply connected region, D, and suppose $u = 0$ on ∂D. Let the constants $\gamma_1, \ldots, \gamma_n$ be defined as in* (6.12). *Suppose that there is another function, v, such that $v_{xy} = 0$ in D, and*

$$v \equiv \beta_i \quad on \quad C_i \qquad (i = 1, \ldots, n), \tag{13}$$

where $\beta_1, \ldots, \beta_n$ are constants. Then

$$\beta_n \gamma_n = \beta_1 \gamma_1 + \cdots + \beta_{n-1} \gamma_{n-1} . \tag{14}$$

Proof. This is an immediate consequence of Green's theorem. If the identity

$$0 = (vu_x)_y - (v_y u)_x$$

is integrated throughout D, since $u = 0$ on ∂D, the result is

$$\int_{C_n} vu_x \, dx = \int_{C_1} vu_x \, dx + \cdots + \int_{C_{n-1}} vu_x \, dx, \tag{15}$$

from which (13) implies (14). ∎

This lemma applies to doubly connected regions, D, whose boundary curves have equations

$$F(x) - G(y) = \beta_i \quad \text{on} \quad C_i \qquad (i = 1, 2), \tag{16}$$

where $\beta_1 \neq \beta_2$. For the annulus, $F(x) = x^2$ and $G(y) = -y^2$. Let

$$v(x, y) \equiv F(x) - G(y) \quad \text{in} \quad D. \tag{17}$$

Then $v_{xy} = 0$ in D, while (13) holds on the boundary. The lemma now implies $\beta_2 \gamma_2 = \beta_1 \gamma_1$. But $\gamma_2 = \gamma_1$ for *every* solution of $u_{xy} = 0$. Since $\beta_1 \neq \beta_2$, we conclude that $\gamma_1 = \gamma_2 = 0$, as in (12).

REFERENCE

1. V. BARCILON, Axi-symmetric inertial oscillations of a rotating ring of fluid, *Mathematika* **15** (1968), 93–102.